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LETTER TO THE EDITOR

An envelope solitary-wave solution for a generalized nonlinear Schrödinger equation

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Abstract. We derive an envelope solitary-wave solution for the following generalized nonlinear Schrödinger equation

$$U_t + \mathrm{i}a_2 U_{xx} + a_3 U_{xxx} = \mathrm{i}b|U|^2 U + b_1(|U|^2 U)_x$$

by gauge transformation.

It is well known that the propagation of optical solitons can be described by the nonlinear Schrödinger (NLS) equation [2]

$$U_t + \mathrm{i}a_2 U_{xx} = \mathrm{i}b|U|^2 U. \tag{1}$$

To improve the transmission rate in optical communications, we should use high-power and ultrashort optical pulses. However, as pulses get shorter and more intense, both higher-order dispersive and nonlinear effects become more important, so we must add some higher-order terms to the NLS equation [1, 3]. The following generalized NLS equation including third-order dispersive (TOD) term and a self-steepening (SS) term can be used to study these effects:

$$U_t + \mathrm{i}a_2 U_{xx} + a_3 U_{xxx} = \mathrm{i}b|U|^2 U + b_1(|U|^2 U)_x \tag{2}$$

where a_2, a_3, b, b_1 are real constants and $a_3 b_1 < 0$.

Optial solitons are also very important for telecommunications, since optical solitons are ideal carriers of information and they can propagate stably over a long distance. However, the number of soliton equations is still limited, since solitons exist only under very special conditions, that is, the dispersive and nonlinear effects must exactly balance.

In this letter we derive an envelope solitary-wave solution for equation (2). This solution has a form similar to that of the envelope one-soliton solution of the NLS equation.

To obtain a travelling-wave solution, consider a gauge transformation

$$U = F(\xi) f(x,t) \tag{G}$$

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where $f(x,t) = \exp \mathrm{i}(kx + mt + \sigma_0)$. $F(\xi)$ is a real function. $\xi = x - vt - \theta_0$, then substituting these expressions into equation (2), we have

$$a_3F'' - (v + 2a_2k + 3a_3k^2)F' - 3b_1F^2F' + \mathrm{i}[(a_2 + 3a_3k)F'' + (m - a_2k^2 - a_3k^3)F - (b + b_1k)F^3] = 0. \tag{3}$$

From the real and imaginary parts of equation (3), we obtain two equations:

$$a_3F''' - (v + 2a_2k + 3_3k^2)F' - 3b_1F^2F' = 0 \tag{4}$$

$$(a_2 + 3a_3k)F'' + (m - a_2k^2 - a_3k^3)F - (b + b_1k)F^3 = 0. \tag{5}$$

Integrating equation (4), we obtain

$$a_3F'' - (v + 2a_2k + 3a_3k^2)F - b_1F^3 = 0. \tag{6}$$

Since F satisfies both equations (5) and (6), we have:

$$\frac{m - a_2k^2 - a_3k^3}{a_2 + 3a_3k} = -\frac{v + 2a_2k + 3a_3k^2}{a_3} = -s \tag{7}$$

$$-\frac{(b + b_1k)}{a_2 + 3a_3k} = -\frac{b_1}{a_3} = r. \tag{8}$$

Then both equations (5) and (6) have the form:

$$F'' - sF + rF^3 = 0. \tag{9}$$

To obtain solitary-wave solutions, we assume $s > 0$, $r > 0$ and integrate equation (9), obtaining

$$(F')^2 = sF^2 - \tfrac{1}{2}rF^4 + C. \tag{10}$$

Letting $C = 0$, and integrating equation (10) again, we have

$$F(\xi) = \sqrt{\frac{2s}{r}}\,\mathrm{sech}(\sqrt{s}\xi).$$

Using equations (7) and (8), we solve k, m, v recursively in terms of s and the coefficients of equation (2), then we obtain the following solution:

$$U = \sqrt{-\frac{2sa_3}{b_1}}\,\mathrm{sech}\left[\sqrt{s}(x - vt - \theta_0)\right] \exp \mathrm{i}(kx + mt + \sigma_0)$$

where s is a positive real number and

$$k = \frac{a_3b - a_2b_1}{2a_3b_1} \tag{11}$$

$$m = -s(a_2 + 3a_3k) + a_2k^2 + a_3k^3 \tag{12}$$

$$v = a_3s - 2a_2k - 3a_3k^2. \tag{13}$$

Since equation (2) resembles an equation considered by Hirota, we show that equation (2) and Hirota's equation are essentially different. If we use the same procedure as above to solve the following equation of Hirota [13]:

$$U_t + \mathrm{i}a_2 U_{xx} + a_3 U_{xxx} = \mathrm{i}b|U|^2 U + 3h|U|^2 U_x \tag{14}$$

then we obtain the following one-soliton solution:

$$U = \sqrt{-\frac{2sa_3}{h}}\,\mathrm{sech}\left[\sqrt{s}(x - vt - \theta_0)\right]\exp \mathrm{i}(kx + mt + \sigma_0)$$

where s is a positive real number and

$$\frac{b + 3hk}{a_2 + 3a_3 k} = \frac{h}{a_3} \tag{15}$$

but v and m have the same form as equations (12) and (13).

k will be eliminated from equation (15) and we get $a_3 b = a_2 h$, this is exactly the constraint condition of Hirota's equation [13].

Comparing the solutions of Hirota's equation and equation (2), we find the following differences between equation (2) and Hirota's equation:

(i) For equation (2), k is completely determined by the coefficients of equation (2), but for Hirota's equation, k is an independent parameter.

(ii) Hirota's equation has a soliton solution under the constraint of $a_3 b = a_2 h$, but equation (2) has a solution under the constraint of $a_3 b_1 < 0$.

(iii) The amplitude of U, $\sqrt{-2sa_3/h}$ is different from $\sqrt{-2sa_3/b_1}$, since b_1 is the coefficient of the term $(|U|^2 U)_x$ and $3h$ is the coefficient of the term $|U|^2 U_x$.

Since equation (2) is obviously different from other soliton equations, I guess that equation (2) could be a new soliton equation, at least we do not have sufficient evidence to exclude this possibility at the moment.

Physically, U_{xxx} is the third-order dispersion (TOD) term, and $(|U|^2 U)_x$ is the self-steepening (SS) term [4–10]. From the existence of the above nice travelling-wave solution, we find that if we add the TOD and SS effects to the NLS equation, these effects may balance again, even the pulse is very short such that we cannot ignore the TOD and SS effects.

We have used split-step Fourier methods [11, 12] to solve equation (2), and the numerical results agree with the above solutions.

Although the envelope solitary-wave solution of equation (2) has a form similar to that envelope one-soliton solution of the NLS equation or Hirota's equation, it is premature to call it a soliton solution, because it is not known whether equation (2) has N-soliton solutions and any other properties of soliton equations. To answer this question, much work needs to be done. Since equation (2) is of direct physical interest, I hope that this work will be done in the future.

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References

[1] Govind Agrawal P 1989 *Nonlinear Fiber Optics* (New York: Academic)
[2] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* SIAM
[3] Kodama Y and Hasegawa A 1987 *IEEE J. Quantum Electron* **QE-23** 510–24
[4] Grischkowsky D, Courtens E and Armstrong J A 1973 *Phys. Rev. Lett.* **31** 422
[5] Tzoar N and Jain M 1981 *Phys. Rev.* A **23** 1266
[6] Anderson D and Lisak M 1983 *Phys. Rev.* A **27** 1393
[7] Yang G and Shen Y R 1984 *Opt. Lett.* **9** 510
[8] Ohkuma K, Ichikawa H and Abe Y 1987 *Opt. Lett.* **12** 512
[9] Kodama Y and Nozaki K 1987 *Opt. Lett.* **12** 1038
[10] Pearson G W, Zanoni R and Krasinski J S 1993 *Opt. Comm.* **103** 507–18
[11] Pathria D and Morris J Ll 1990 *J. Comput. Physics* **87** 108–25
[12] Pathria D and Morris J Ll 1989 *Phys. Scr.* **39** 673–9
[13] Hirota R 1973 *J. Math. Phys.* **14** 805–9